What you as a future teacher need to know about fractions and why

November 8, 2016

Contents

One

How to read these notes

If you just care about to learn what you need to know about fractions in this course, you can jump to chapter 3. But if if you need some motivation to work on your understanding of fractions and can't see what understanding can give you and your future students, then start with chapter 2.

The numbering of problems starts over on each page. This way, it's easy to find them when we refer to them. For example, problem 8.1 is on page 8.

Two

Why bother with understanding fractions?

The difficulty in this Math 1480 course is to deal with the fact that quick methods we learned to add, multiply and divide fractions need to be understood. So you learned to do the following: you multiply fractions by multiplying the numerators and denominators

$$\begin{aligned}\frac{2}{7}\cdot\frac{3}{4} &= \frac{2\cdot 3}{7\cdot 4}\\ &= \frac{6}{28}\end{aligned}$$

and you add fractions by taking a common denominator

$$\begin{aligned}\frac{2}{7}+\frac{3}{4} &= \frac{2\cdot 4}{7\cdot 4}+\frac{3\cdot 7}{4\cdot 7}\\ &= \frac{8}{28}+\frac{21}{28}\\ &= \frac{8+21}{28}\\ &= \frac{29}{28}\end{aligned}$$

and you divide fractions by multiplying by the reciprocal of the divisor

$$\begin{aligned}\frac{2}{7}\div\frac{3}{4} &= \frac{2}{7}\cdot\frac{4}{3}\\ &= \frac{2\cdot 4}{7\cdot 3}\\ &= \frac{8}{21}.\end{aligned}$$

I know that many of you think this is all that need to be known about fractions. You express this time after time, by saying stuff like "*Why do we have to bother with all these complicated explanations and pictures when I can do all this much faster by using the methods in red above.*"

No doubt, kids don't have to know much more about fractions to pass most of the tests they are given—especially the year-end TNREADY tests. But kids and hence their teachers need to *understand* mathematics, not just be able to calculate. Most people don't ever have to add or multiply fractions in their lives, so why would anybody think, drilling kids with these calculations is important?!

Well, it's *not* very important at all. But understanding fractions *is* important. There are two related reasons for it.

Understanding fractions is needed later Without understanding fractions, you can't comprehend numbers which are not integers, you can't understand percentages, interest rates, the meaning of decimal expressions, trigonometry, statistics, and even the simplest laws of physics, chemistry, economics or art. In these practical matters, being able to add or multiply fractions is unimportant. What is important is to understand the *meaning* of fractions. Can you understand Newton's law

$$\frac{F}{a} = m$$

without understanding fractions? Can you understand what the sine of an angle *really* is if it's given as

$$\sin\alpha = \frac{a}{c}$$

without understanding fractions?

Now, if *you* don't understand fractions, how will your students understand them? How will they be prepared for high school not to mention college? The fact is that fractions are at the heart of mathematics and hence they pop up everywhere even where you don't expect them, like in art or music.

Learn to ask why The real *purpose* of math is not about teaching people how to calculate complicated things using weird formulas. Most people don't ever have to calculate anything complicated in their jobs. How many people do you know who ever have to deal with

$$\frac{\frac{2}{7} - \frac{3}{8}}{\frac{2}{5}}?$$

On the other hand, it is useful for everybody to learn to ask "*why*". I am not just talking about asking "Mom, why do I have to be home by midnight?" or "Why are you mad at me?", though these questions also fit into the general idea: we want to understand why things happen, and once we get a good explanation, we are happier or at least we are at peace. Mathematics is the main way of teaching kids to ask questions that will help them to understand things: "Why do motor bikes accelerate better than cars?" or "Why are some twins similar to each other and some aren't?" or "Why do I have to slow my car down on a bridge?" or "Why do I like the scent of roses, but don't want to eat it?" or "Why aren't there any hurricanes or tornadoes in Europe?" or "Why is the Earth round and not shaped like a cube?" or "Why do people like bitter beer and sweet chocolate but not many like sweet beer and bitter chocolate?".

Mathematics is *the* subject where kids can learn to ask why things work and can also learn to search for answers and come up with answers.

It is one thing to make a kid learn to add fractions by taking common denominators, but it's a completely different experience to see them light up and say "So that's what common denominator is!". You can be sure, they will not forget that moment of enlightment, while they surely will forget how to add fractions. Is it worth spending a lot of time on something you later forget without a trace?

So remember: explaining *why* things are comes before and always more difficult than explaining *how* things are.

I certainly feel this as I am trying to explain to you *why* bother with learning about fractions. ☺

Three

Understanding fractions: what you need to know

3.1 What is $\frac{1}{2}$ of something?

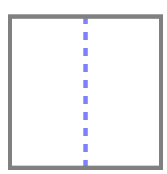
What is the meaning of $\frac{1}{2}$ of something? *It means, divide that something up into 2* ***equal*** *pieces and we say that one (any) of these pieces is "$\frac{1}{2}$" of the original object.*

Kinda a complicated definition, and there is no reason to memorize it. Just the process of *dividing up* or *cutting up* something into *equal* pieces needs to be hammered in, and that's done via examples.

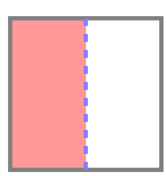
Initially, when you teach the concept, you should talk about the half of all kinds of things, like half of a pizza, a dollar, an apple, 6 M&Ms. But when we explain the properties of fractions, you should stick to two models, and use them over and over, because this helps kids translate real life objects and problems into mathematics—the most difficult and important part of solving word problems.

The two basic models we use for fractions is a rectangle (mostly square) and a line segment. The square (rectangle) is the most useful way of visualizing fractions while the line segment is the most important since it helps kids identify numbers on their standard model, the number line.

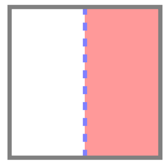
Visualizing $\frac{1}{2}$ of a square is what you'd expect: divide the square up into two equal pieces



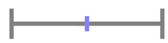
and take one of the pieces



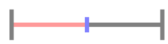
Of course, you can take the other half piece, if you want,



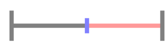
To introduce our other standard model, the line segment, and to visualize $\frac{1}{2}$ of it, divide it up into 2 equal pieces



and take one piece

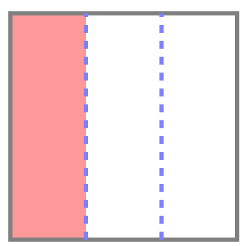


You can take the other piece, if you want



Problem 8.1. Illustrate $\frac{1}{3}$, using both the square and interval models, by cutting them up into 3 equal pieces.

Answer. Square model



Line model:

□

Problem 8.2. Illustrate $\frac{1}{10}$, using both the square and interval models, by cutting them up into 10 equal pieces.

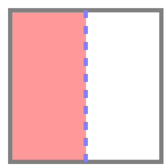
What have we learned in section 3.1?

In this section 3.1 we have learned the following: take any object which we denote by x. Taking $\frac{1}{b}$ of x means dividing x into b *equal* pieces and then taking one of these pieces. The one piece we take can be *any* of the pieces.

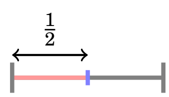
3.2 What is $\frac{1}{2}$?

Now that we know what "$\frac{1}{2}$ of something" is we need to understand what $\frac{1}{2}$ *as a number* is. What do I mean by this? Well, there is difference between talking about 6 apples or just about the number 6. Similarly, there is a difference between talking about $\frac{1}{2}$ of an apple or just about the number $\frac{1}{2}$. The 6 in "6 apples" refers to the quantity of the apples. So $\frac{1}{2}$ in "$\frac{1}{2}$ of an apple" refers to the quantity or the size of the apple we get after cutting it into two equal parts.

What is then the meaning of $\frac{1}{2}$ in terms of the size of a square? The size of a square is its *area* hence if the area of the square below is 1, then the area of the pink rectangle is $\frac{1}{2}$



The size of a line segment is its *length* hence if the length of a line segment below is 1, then the length of the pink segment below is $\frac{1}{2}$



Problem 9.1. Explain the meaning of $\frac{1}{3}$ as the area of one piece after dividing the square of area 1 into 3 equal pieces.

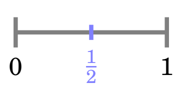
Problem 9.2. Explain the meaning of $\frac{1}{3}$ as the length of one piece after dividing the line segment of length 1 into 3 equal pieces.

Problem 9.3. Explain the meaning of $\frac{1}{10}$ as the area of one piece after dividing the square of area 1 into 10 equal pieces.

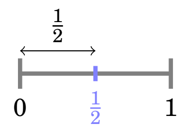
Problem 9.4. Explain the meaning of $\frac{1}{10}$ as the length of one piece after dividing the line segment of length 1 into 10 equal pieces.

The importance of the line segment illustration is that if the line segment is the unit interval

then the midpoint of the interval *is* $\frac{1}{2}$,



Let's not forget that the reason for calling the midpoint $\frac{1}{2}$ is because the length of the midpoint from the origin is $\frac{1}{2}$



Problem 10.1. Illustrate $\frac{1}{3}$ both as a length and a point in the unit interval.

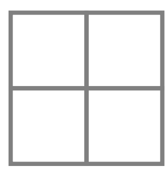
Problem 10.2. Illustrate $\frac{1}{10}$ both as a length and a point in the unit interval.

What have we learned in section 3.2?

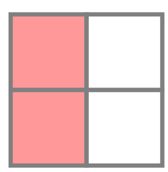
In this section 3.2 we have learned that there is a difference between taking $\frac{1}{b}$ of any object x and $\frac{1}{b}$ *as a number*. The number $\frac{1}{b}$ refers to the *quantity* or *size* we take and not the actual piece of x after subdividing it into b equal parts. The relationship between the number "$\frac{1}{b}$" and "taking $\frac{1}{b}$ of x" is the same as the relationship between the number "6" and "taking 6 of x". If the quantity of the object x is equal 1, then the quantity of $\frac{1}{b}$ of x is equal $\frac{1}{b}$.

3.3 CONNECTION BETWEEN $\frac{1}{2}$ AND DIVISION BY 2

Let us take a square of side length 2 so of area $2 \cdot 2 = 4$



The area of each smaller square is 1. Now let us take $\frac{1}{2}$ of the big square



Since the pink rectangle contains 2 of the small squares, its area is 2. This tells us that taking $\frac{1}{2}$ of 4 is 2, as if we *divided* 4 by 2.

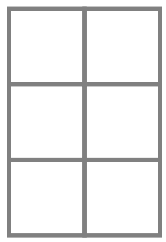
$$\frac{1}{2} \text{ of } 4 \text{ is the same as } 4 \div 2.$$

Problem 10.3. Illustrate, using the interval model, that $\frac{1}{2}$ of 4 is the same as dividing 4 by 2.

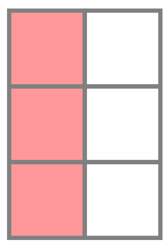
Hint. Start with a line segment of length 4 such as

□

Let us now take a rectangle of area 6



and take half of this rectangle



Since the pink rectangle contains three of the small squares (each of which still has area 1), its area is 3. Dividing 6 by 2 we also get 3, so we have

$$\tfrac{1}{2} \text{ of } 6 \text{ is the same as } 6 \div 2.$$

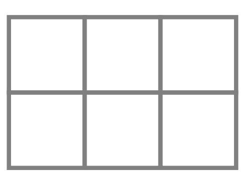
Problem 11.1. Illustrate, using the interval model, that $\frac{1}{2}$ of 6 is the same as dividing 6 by 2.

Hint. Start with a line segment of length 6 such as

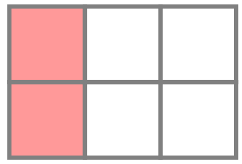
□

Problem 11.2. Illustrate, using both the rectangle and line segment models, that $\frac{1}{3}$ of 6 is the same as dividing 6 by 3.

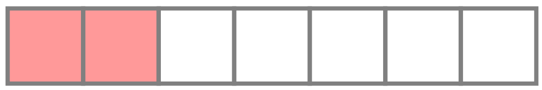
Answer. For the rectangle model, one possibility is to consider this 2×3 rectangle



and take $\frac{1}{3}$ of it



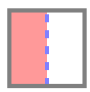
Of course, you could have taken other rectangles of area 6 like a 1×6 one



$\square$

The difference between taking $\frac{1}{2}$ of an integer and dividing that integer by 2 is that we can *always* take the $\frac{1}{2}$ of an integer while we can divide by 2 only those integers which are multiples of 2. For odd numbers, we get a remainder.

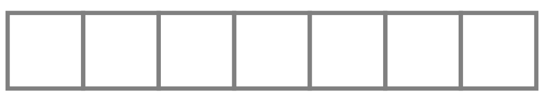
The simplest case is 1: $\frac{1}{2}$ of 1 is, well, $\frac{1}{2}$,



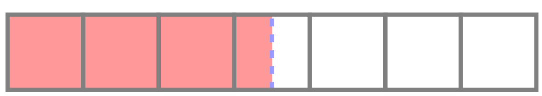
while dividing 1 by 2 gives us a remainder

$$1 = 0 \cdot 2 + 1.$$

This was a bit singular case. To illustrate the general case, take 7: taking $\frac{1}{2}$ of 7 can be illustrated by considering this rectangle of area 7



and then taking $\frac{1}{2}$ of it

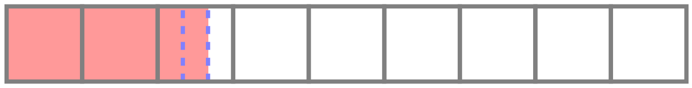


The area of the pink rectangle is 3 plus $\frac{1}{2}$ since it takes out the left half of the middle square. On the other hand, when we divide 7 by 2 we get 3 as the quotient and 1 as the remainder

$$7 = 3 \cdot 2 + 1.$$

Problem 12.1. Illustrate, using the rectangle model, the difference between taking $\frac{1}{3}$ of 8 and dividing 8 by 3.

Answer. Illustrating $\frac{1}{3}$ of 8 on a 1×8 rectangle, you'd get



which means 2 full squares and 2 $\frac{1}{3}$ pieces from the third square. On the other hand, when you *divide* 8 by 3 you get 2 as a remainder

$$8 = 2 \cdot 3 + 2.$$

□

What have we learned in section 3.3?

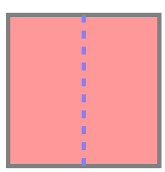
In this section 3.3 we have learned that there is a connection between *division* of a number a by b and taking $\frac{1}{b}$ of a. The two operations are the *same* if a is a multiple of b: if $a = bq$ then $a \div b = q$ and $\frac{1}{b}$ of a is also the quotient q. The two operations are *different* if a is not a multiple of b. In this case, we *cannot* divide a by b, so $a \div b$ is not defined because we get a *remainder* r when we try to perform the division, so all we write is $a = b \cdot q + r$. On the other hand, $\frac{1}{b}$ of a is *always* defined: it is equal with $a \cdot \frac{1}{b}$, that is, it's equal with the quantity we get when we take a copies of $\frac{1}{b}$.

3.4 $\frac{3}{2}$ IS A SHORTHAND FOR $3 \cdot \frac{1}{2}$

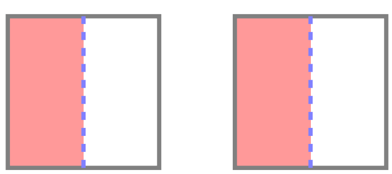
Once we understand what $\frac{1}{2}$ is, we also have to understand the notation $\frac{2}{2}$, $\frac{3}{2}$, $\frac{4}{2}$, etc. Well they are just a *shorthand* for $2 \cdot \frac{1}{2}$ and $3 \cdot \frac{1}{2}$, $4 \cdot \frac{1}{2}$, etc. respectively,

$$\frac{2}{2} = 2 \cdot \frac{1}{2} \quad \text{and} \quad \frac{3}{2} = 3 \cdot \frac{1}{2} \quad \text{and} \quad \frac{4}{2} = 4 \cdot \frac{1}{2}.$$

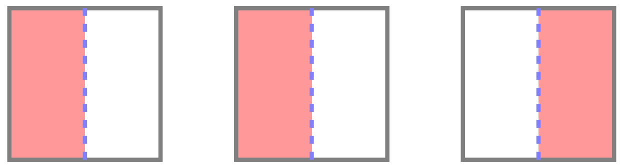
So $\frac{2}{2}$ means taking two $\frac{1}{2}$ of something. Using squares for illustration, we could take the two $\frac{1}{2}$ pieces from the same square



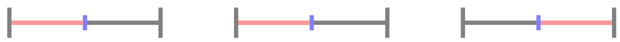
or we can take them from two different squares



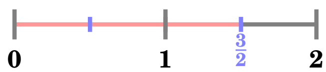
Similarly, when we want to illustrate $\frac{3}{2}$, we could take the three $\frac{1}{2}$ pieces from 3 different squares



We can illustrate $\frac{3}{2}$ via line segments similarly to the above



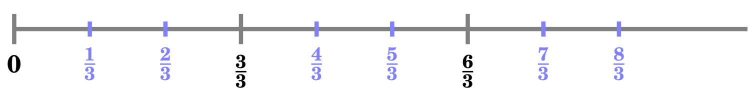
It is much more usual, though, to take the three $\frac{1}{2}$ next to each other, to get the familiar numberline picture



After all, it's immaterial where we got the line segments of length $\frac{1}{2}$ from.

Problem 14.1. Mark the first eight $\frac{1}{3}$ on the numberline, starting at 0.

Answer. Here is one way to do it



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What have we learned in section 3.4?

In this section 3.4 we have learned that $\frac{a}{b}$ is a shorthand for $a \cdot \frac{1}{b}$. It is important to realize that $\frac{a}{b}$ is *defined* with the help of $a \cdot \frac{1}{b}$ and not the other way around.

3.5 Adding fractions: same denominator, $\frac{2}{2} + \frac{3}{2} = \frac{2+3}{2}$

We of course calculate $\frac{2}{2} + \frac{3}{2}$ as

$$\begin{aligned}\frac{2}{2} + \frac{3}{2} &= \frac{2+3}{2} \\ &= \frac{5}{2},\end{aligned}$$

and eventually this is exactly how the kids are going to do this. But we have to understand *why* we can do it this way.

The purely mathematical reason presents itself as soon as we recall, from section 3.4, that $\frac{2}{2}$ is a short hand for $2\cdot\frac{1}{2}$ and $\frac{3}{2}$ is a short hand for $3\cdot\frac{1}{2}$. Then we can calculate, using our favorite identities, as

$$\frac{2}{2}+\frac{3}{2}=2\cdot\frac{1}{2}+3\cdot\frac{1}{2}$$

which, by the distributive law,

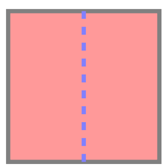
$$=(2+3)\cdot\frac{1}{2}$$
$$=5\cdot\frac{1}{2}$$

which, since $5\cdot\frac{1}{2}$ is a shorthand for $\frac{5}{2}$,

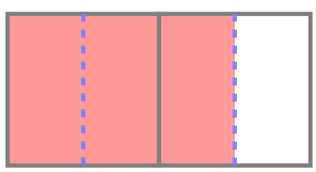
$$=\frac{5}{2}.$$

Many of you didn't like this lengthy algebra, so let's do the same thing but with illustration.

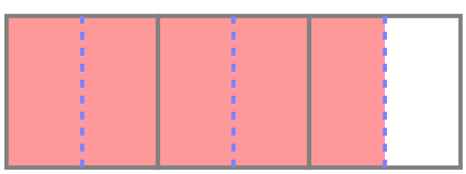
So let us first take $\frac{2}{2}$, that is two $\frac{1}{2}$ pieces



Let us also take $\frac{3}{2}$, which is three $\frac{1}{2}$ pieces



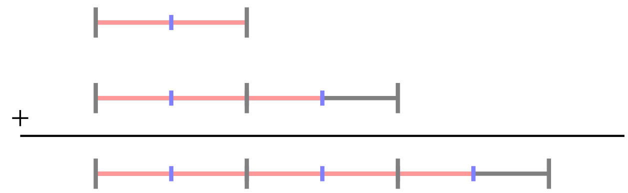
Since adding means collecting all the pieces we marked, we collect the two and three $\frac{1}{2}$ pieces and put them all into one place



We see five $\frac{1}{2}$ pieces, which, in shorthand is $\frac{5}{2}$.

Problem 15.1. Use the line segment model to illustrate $\frac{2}{2}+\frac{3}{2}$

Answer.



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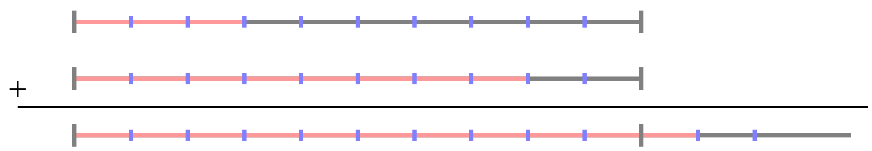
We start feeling that the line segment model is more economical than the square one. This feeling just gets stronger as we take fractions with larger denominators.

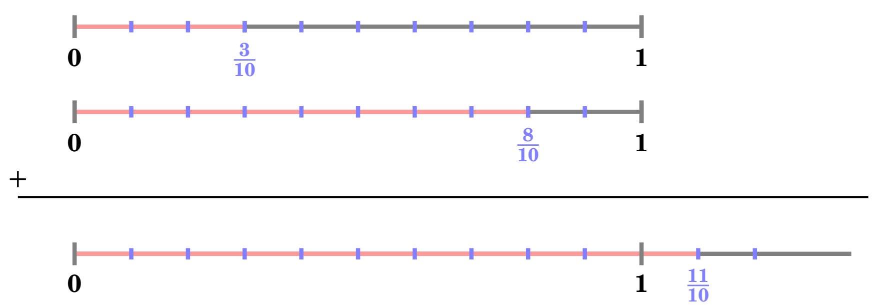
Problem 16.1. Use both the square and line segment model to illustrate $\frac{3}{10} + \frac{8}{10}$.

Answer. I only give the answer in terms of the line segment model.



□

Even the line segment solution to this last problem, problem 16.1, was not easy to follow. As a minimum, we should indicate the numbers $\frac{3}{10}, \frac{8}{10}, \frac{11}{10}$ on our picture, as in



This is the point where I stop using squares or rectangles for illustration in these notes. In a class you may continue to use them—depending on how the kids feel about using line segments exclusively. If they start to get confused, keep the rectangles for a while longer.

What have we learned in section 3.5?

In this section 3.5 we have learned that $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

3.6 Multiplying a fraction by an integer, $2\cdot\frac{3}{2}=\frac{2\cdot 3}{2}$

We of course calculate $2\cdot\frac{3}{2}$ as

$$\begin{aligned} 2\cdot\frac{3}{2} &= \frac{2\cdot 3}{2} \\ &= \frac{6}{2}. \end{aligned}$$

and eventually this is exactly how the kids are going to do this. But we have to understand *why* we can do it this way.

The purely mathematical reason is based on that $\frac{3}{2}$ is a short hand for $3\cdot\frac{1}{2}$. Then we can calculate as

$$2\cdot\frac{3}{2} = 2\cdot\left(3\cdot\frac{1}{2}\right)$$

which, by the associative law, is

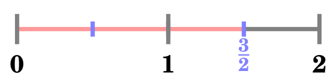
$$\begin{aligned} &= (2\cdot 3)\cdot\frac{1}{2} \\ &= 6\cdot\frac{1}{2} \end{aligned}$$

which, since the shorthand for $6\cdot\frac{1}{2}$ is $\frac{6}{2}$, is

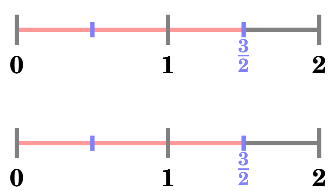
$$= \frac{6}{2}.$$

No doubt, this is dry algebra, so let's do the same thing but with illustration.

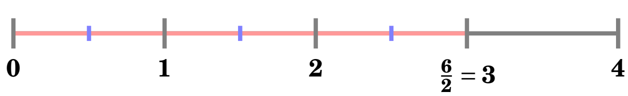
Let us first take $\frac{3}{2}$, that is, 3 pieces of $\frac{1}{2}$



Multiplying anything by 2 means we have to repeat the same thing twice



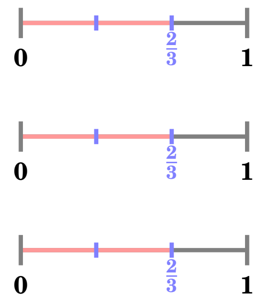
We now have to collect the 2 pieces of pink lines of length $\frac{3}{2}$ is one place. We get



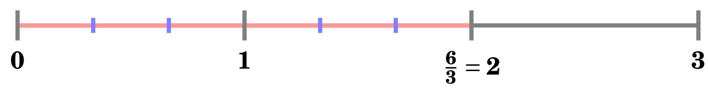
that is, $2\cdot 3 = 6$ pieces of $\frac{1}{2}$.

Problem 17.1. Illustrate the multiplication $3 \cdot \frac{2}{3}$.

Answer. $3 \cdot \frac{2}{3}$ means taking 3 identical copies of $\frac{2}{3}$



Now collect the 3 pieces of line segments of length $\frac{2}{3}$,



which, as you see, is 6 pieces of $\frac{1}{3}$, that is, $\frac{6}{3} = 2$. □

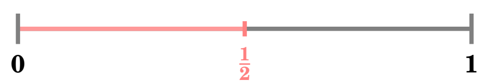
Problem 18.1. Illustrate the multiplication $4 \cdot \frac{8}{10}$

What have we learned in section 3.6?

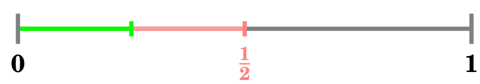
In this section 3.6 we have learned that $c \cdot \frac{a}{b} = \frac{c \cdot a}{b}$.

3.7 WHY IS $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$?

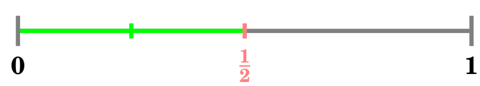
Let us look at an "enlarged" version of $\frac{1}{2}$



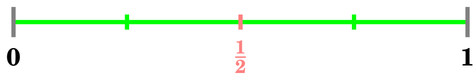
Divide the pink line segment into 2 equal parts with a green tick and also color the left interval green



What is the length of the green line segment? Well, 2 of them put after each other would fill the pink segment



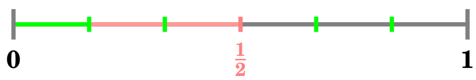
Since two of the pink line segments filled the whole interval, $2 \cdot 2 = 4$ of the green line segments would fill the whole interval.



This means that the length of the green interval is $\frac{1}{4}$, and hence $\frac{1}{2} = \frac{2}{2 \cdot 2} = \frac{2}{4}$.

Problem 19.1. Show that $\frac{1}{2} = \frac{3}{6}$ by showing that dividing a line segment of length $\frac{1}{2}$ into 3 equal parts, the length of one part will be $\frac{1}{6}$.

Hint. The length of the green line segment below is $\frac{1}{3}$ of $\frac{1}{2}$, hence $3 \cdot 2 = 6$ of them will fill the whole $[0, 1]$ interval, as the green ticks can help you counting this out.



□

Problem 19.2. Show that $\frac{1}{3} = \frac{2}{6}$ by showing that dividing a line segment of length $\frac{1}{3}$ into 2 equal parts, the length of one part will be $\frac{1}{6}$.

Problem 19.3. Show that $\frac{1}{3} = \frac{3}{9}$ by showing that dividing a line segment of length $\frac{1}{3}$ into 3 equal parts, the length of one part will be $\frac{1}{9}$.

What have we learned in section 3.7?

In this section 3.7 we have learned that $\frac{1}{b} = \frac{c}{c \cdot b}$.

3.8 Why is $\frac{3}{2} = \frac{6}{4} = \frac{9}{6}$?

We already know in case of a fraction of the form $\frac{1}{b}$, like $\frac{1}{2}$, why we can multiply both the top and the bottom by the same number, and the fraction doesn't change,

$$\frac{1}{2} = \frac{2}{2 \cdot 2} = \frac{3}{2 \cdot 3} = \frac{4}{2 \cdot 4}.$$

How about an arbitrary fraction like $\frac{3}{2}$, so when the top is not equal 1? Why do we have

$$\frac{3}{2} = \frac{3 \cdot 2}{2 \cdot 2} = \frac{3 \cdot 3}{2 \cdot 3} = \frac{3 \cdot 4}{2 \cdot 4}?$$

The argument is, actually, simple algebra once we recall that $\frac{a}{b}$ is just a shorthand for $a \cdot \frac{1}{b}$. For example

$$\frac{3}{2} = 3 \cdot \frac{1}{2}$$

since we know that $\frac{1}{2} = \frac{2}{2 \cdot 2}$

$$= 3 \cdot \frac{2}{2 \cdot 2}$$

since in section 3.6 we learned how to multiply a fraction by an integer

$$= \frac{3 \cdot 2}{2 \cdot 2}.$$

Problem 20.1. Using the steps in the text, explain why $\frac{3}{2} = \frac{3 \cdot 4}{2 \cdot 4}$

Problem 20.2. Using the steps in the text, explain why $\frac{2}{3} = \frac{14}{21}$, that is, why $\frac{2}{3} = \frac{2 \cdot 7}{3 \cdot 7}$.

Problem 20.3 (Simplifying a fraction)**.** *Reversing* the steps in the text, explain why $\frac{12}{20} = \frac{3}{5}$, that is, why $\frac{3 \cdot 4}{5 \cdot 4} = \frac{3}{5}$.

Answer.

$$\begin{aligned} \frac{3 \cdot 4}{5 \cdot 4} &= 3 \cdot \frac{4}{5 \cdot 4} \\ &= 3 \cdot \frac{1}{5} \\ &= \frac{3}{5} \end{aligned}$$

□

Problem 20.4. Explain why $\frac{77}{121} = \frac{7}{11}$.

What have we learned in section 3.8?

In this section 3.8 we have learned that $\frac{a}{b} = \frac{c \cdot a}{c \cdot b}$, that is, we can multiply both the top and the bottom of a fraction $\frac{a}{b}$ by the same number c, without changing the fraction. Note that when we read this backwards, so $\frac{c \cdot a}{c \cdot b} = \frac{a}{b}$, then this tells us that a fraction $\frac{c \cdot a}{c \cdot b}$ can be *simplified* by the factor c to get $\frac{a}{b}$. In other words, we can *divide* both the top and the bottom of a fraction by the same number, without changing it.

3.9 Adding fractions of any denominators, $\frac{1}{2} + \frac{1}{3}$

To be done

What have we learned in section 3.9?

In this section 3.9 we have learned the following two step procedure to add the two fractions $\frac{a}{b}$ and $\frac{c}{d}$:

1. Convert fractions to have the same denominator $b \cdot d$.

$$\begin{aligned} \frac{a}{b} &= \frac{a \cdot d}{b \cdot d} \\ \frac{c}{d} &= \frac{c \cdot b}{d \cdot b}. \end{aligned}$$

2. Add the converted fractions. This is now easy since they have the same denominator so we can use what we learned in section 3.5,

$$\frac{a\cdot d}{b\cdot d}+\frac{c\cdot b}{d\cdot b}=\frac{a\cdot d+c\cdot b}{d\cdot b}.$$

In practice, the steps are done in a single sequence of two equations

$$\begin{aligned}\frac{a}{b}+\frac{c}{d}&=\frac{a\cdot d}{b\cdot d}+\frac{c\cdot b}{d\cdot b}\\&=\frac{a\cdot d+c\cdot b}{d\cdot b}.\end{aligned}$$

3.10 $\frac{1}{2}$ of x is the same as $\frac{1}{2}\cdot x$

To be done

What have we learned in section 3.10?

In this section 3.10 we have learned that “taking $\frac{a}{b}$ of x” means simply $\frac{a}{b}\cdot x$.

3.11 Multiplying fractions, $\frac{3}{2}\cdot\frac{5}{3}=\frac{3\cdot 5}{2\cdot 3}$

To be done

What have we learned in section 3.11?

In this section 3.10 we have learned that $\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$.

3.12 Dividing fractions, $\frac{3}{2}\div\frac{5}{3}=\frac{3}{2}\cdot\frac{3}{5}$

To be done

What have we learned in section 3.12?

In this section 3.12 we have learned that $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{a\cdot d}{b\cdot c}$.

3.13 Loose ends: improper and mixed fractions

To be done

What have we learned in section 3.13?

In this section 3.13 we have learned that an *improper* fraction is a fraction $\frac{a}{b}$ where $a\geq b$, and a *mixed* fraction is of the form $q\frac{r}{b}$, where q is an integer and $\frac{r}{b}$ is a *proper* fraction, that is $r<b$. To convert an improper fraction $\frac{a}{b}$ to its mixed form, divide a by b and note the remainder, $a=b\cdot q+r$, and then you get that $\frac{a}{b}=q\frac{r}{b}$. To convert a mixed fraction $q\frac{r}{b}$ to its improper form, set calculate a from $a=b\cdot q+r$ and then you will have $q\frac{r}{b}=\frac{a}{b}$.

3.14 Word problems

To be done

What have we learned in section 3.14?

In this section 3.14 we have learned that the main difficulty in word problems is the translation of the problem to mathematics. To help this translation, pictures should be used. The pictures should be chosen carefully so that just by looking at them, it triggers the correct mathematics to solve the problem. Initially, the pictures can and should mimic the “story” in the problem, so you can use pizzas, shoes, eggs. But the diagrams should become simpler with less and less specific contents. In case of word problems involving fractions, the best translational tools are the rectangles and line segments. Since the most straight forward connection is between line segments and numbers, eventually, the diagrams should be line segments.