Math PhD Qualifying Exam | **Analysis 2020 Fall** | **Last Name:** ____________ **First Name:** ____________

- This exam has 8 questions, and some questions have 2 parts. In case a question has 2 parts, you are supposed to solve both parts to get full credit for the problem.
- *Choose any 5 of the 8 questions below to solve. You have the option to submit a solution to more than 5 problems, but you need specify which 5 we should grade.*
- The Lebesgue measure on $\mathbb{R}$ or on a subset of $\mathbb{R}$ is denoted by λ.
- Deadline to submit: 13:30, August 15, 2020.

1. Show that $L^2(\mathbb{R})$ is a complete metric space, that is, every Cauchy sequence is convergent. (You only have to show the completeness of this metric space, so you can assume that we already know that it's a metric space. Recall that the norm $\|f\|$ in $L^2(\mathbb{R})$ is defined by the formula $\|f\| := \left(\int_{\mathbb{R}} |f|^2\right)^{1/2}$ and the distance in $L^2(\mathbb{R})$ is $\mathrm{d}(f,g) := \|f-g\|$.)

2. Consider the measure space $([0,1],\lambda)$, so the unit interval with the Lebesgue measure λ. In particular, we have $\lambda([0,1]) = 1$.
   (a) Show that $L^2[0,1] \subset L^1[0,1]$.
   (b) Give an example for a function f which is in $L^1[0,1]$ but not in $L^2[0,1]$.

3. Consider $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$.
   (a) Show that $L^1(\mathbb{R}) \not\subset L^2(\mathbb{R})$.
   (b) Show that $L^2(\mathbb{R}) \not\subset L^1(\mathbb{R})$.

4. Let (f_n) be a sequence of orthonormal elements in an inner product space $(X,(f,g))$, that is, $\|f_n\| = 1$ for every $n \in \mathbb{N}$ and $(f_n,f_m) = 0$ for $n \neq m$.

   Show that then $\lim_N \left\| \frac{1}{N} \sum_{n \le N} f_n \right\| = 0$.

   (Recall that $\|f\| := \sqrt{(f,f)}$.)

5. (a) Give an example for a sequence (f_n) of $L^1(\mathbb{R})$ functions which converges to 0 almost everywhere but it doesn't converge to 0 in L^1-norm.
   (b) Give an example for a sequence (f_n) of $L^1(\mathbb{R})$ functions which converges to 0 in L^1-norm but not almost everywhere.

6. For $f,g \in L^1(\mathbb{R})$, define their *convolution* $f * g$ by $(f*g)(x) := \int_{y\in\mathbb{R}} f(x)g(x+y)$.

   Show that $\|f*g\|_{L^1(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})}\|g\|_{L^1(\mathbb{R})}$, that is,

$$\int_{\mathbb{R}} |f*g| \le \int_{\mathbb{R}} |f| \cdot \int_{\mathbb{R}} |g|.$$

7. (a) Let (T_n) be a sequence of linear $L^1(\mathbb{R}) \to L^1(\mathbb{R})$ operators so that $\|T_n f\|_{L^1(\mathbb{R})} \le \|f\|_{L^1(\mathbb{R})}$ for every $f \in L^1(\mathbb{R})$ and every n. Suppose that there is a dense set $D \subset L^1(\mathbb{R})$ so that $\lim_{n\to\infty} T_n f = f$ in $L^1(\mathbb{R})$-norm for every $f \in D$.

   Show that then $\lim_{n\to\infty} T_n f = f$ in $L^1(\mathbb{R})$-norm for every $f \in L^1(\mathbb{R})$.

(b) Let (T_n) be a sequence of linear $L^1(\mathbb{R}) \to L^1(\mathbb{R})$ operators so that

$$\lambda\{\, x \mid x \in \mathbb{R}, \sup_n |T_n f(x)| > \alpha \,\} \leq \frac{1}{\alpha} \cdot \|f\|_{L^1(\mathbb{R})}$$

for every $f \in L^1(\mathbb{R})$ and every $\alpha > 0$. Suppose that there is a dense set $D \subset L^1(\mathbb{R})$ so that $\lim_{n\to\infty} T_n f = f$ almost everywhere for every $f \in D$.

Show that then $\lim_{n\to\infty} T_n f = f$ almost everywhere for every $f \in L^1(\mathbb{R})$.

8. Let (U_n) be a sequence of open subsets of $\mathbb{R}$. Assume that for each n, the set U_n is (everywhere) dense in $\mathbb{R}$.

   Show that then the set $\cap_n U_n$ is nonempty.

End of exam.